Approximate solutions of two-way diffusion equations

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In this paper, a general and systematic scheme is formulated for finding approximate solutions of two-way diffusion equations. This expansion scheme is valid for arbitrary mean-free path and can be carried out to any desired accuracy. Its potential is demonstrated by constructing approximate solutions for two problems concerning the kinetics of an electron beam, and the accuracy is found to be very good even when only a few terms are included in the expansion. The approximate solutions found are compared with numerical calculations and previous analytical work in the literature.

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I. INTRODUCTION

Kinetic equations for gases and plasmas are notoriously difficult to solve in general. Analytical techniques are rarely applicable unless the collisional mean-free path is much shorter or much longer than the scale length of the system. Instead, one often resorts to Monte Carlo simulations to determine transport properties of gases and plasmas in the intermediate mean-free-path regime. In this paper, we present an alternative, analytical method for finding approximate solutions to kinetic problems that are of the form of two-way diffusion equations [1]. Such equations often occur in kinetic plasma theory, where the long range of Coulomb interaction implies diffusive spreading of the distribution function in velocity space. The kinetic equation, therefore, resembles a diffusion equation, but one where the direction of ''time'' is different in different parts of velocity space, i.e., a two-way diffusion equation.

Our analytical approximation scheme relies on an expansion in certain eigenfunctions. As explained in Refs. [1,2], the eigenfunctions for two-way diffusion equations are of somewhat unconventional nature (Sturm-Liouville theory is not applicable), and this is what makes the solution difficult. Section II contains a general discussion of two-way diffusion equations followed by a description of the expansion technique. This scheme is then applied to two different problems concerning the kinetics of electron beams in Sec. III and Sec. IV. The first of these problems is that of electron transport in a scattering medium, with applications to, e.g., electron transport in plasmas and deposition of pencil beams in radiotherapy [3,4]. The second problem is one that recently surfaced in the theory of damping of relativistic electron beams by the emission of synchrotron radiation, which has applications to runaway electron currents in tokamaks [5]. In both these problems, it is found that good accuracy is obtained by including only a few terms in the expansion. Our conclusions are summarized in Sec. V.

II. EXPANSION PROCEDURE

Following the work by Fisch and Kruskal [1], we consider the general two-way diffusion equation in one dimension,

$$h(\mu)\frac{\partial f(x,\mu)}{\partial x}=\frac{\partial}{\partial \mu}D(\mu)\frac{\partial f(x,\mu)}{\partial \mu}=\mathcal{L}(f). \tag{1}$$

We assume that $D(\mu)$ is positive in the interval $a\leqslant\mu\leqslant b$ but allow $h(\mu)$ to change sign at least once. This means that diffusive spreading in μ takes place with increasing x in the region where h is positive, and with decreasing x where h is negative. In other words, the arrow of ''time'' is in opposite directions in different regions. The domain under consideration is $a\leqslant\mu\leqslant b$ and $0\leqslant x\leqslant L$, and the problem is well-posed when boundary conditions are prescribed on the entire boundaries $\mu=a$ and $\mu=b$, but only on parts of the boundaries $x=0$ and $x=L$. At $x=0$, ''initial'' conditions should be specified where h is positive, and at $x=L$ ''final'' conditions should be given where h is negative,

$$f(0,\mu)=f_+(\mu),\quad h(\mu)>0,$$

$$f(L,\mu)=f_-(\mu),\quad h(\mu)<0.$$

If x denotes a space coordinate, this corresponds to specifying the distribution of the particles that enter the medium, while the equation itself determines the distribution of emitted particles.

If a conventional separation of variables is attempted, one obtains the eigenvalue problem

$$\mathcal{L}(f_n)=-\lambda_n h(\mu)f_n(\mu), \tag{2}$$

where the integer n takes values $-\infty<n<\infty$. It is clear that

$$f(x,\mu)=f_n(\mu)e^{-\lambda_n x}, \tag{3}$$

is a solution to Eq. (1) if the boundary conditions are ignored. In a conventional Sturm-Liouville problem, the solution could then be expressed as a linear combination of such eigensolutions. For, when typical boundary value problems are solved by separating variables, Sturm-Liouville theory guarantees that the eigenfunctions form a complete and orthogonal system. However, since h changes sign in the interval of interest ($a<\mu<b$) conventional Sturm-Liouville theory does not apply, and the eigenfunctions do not necessarily form a complete system. Nevertheless, the system be-

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comes complete if it is supplemented by an additional function, $g(\mu)$, that satisfies the equation

$$\frac{\partial}{\partial\mu}D(\mu)\frac{\partial g(\mu)}{\partial\mu}+h(\mu)=0, \tag{4}$$

if such a function exists. (Otherwise the eigenfunctions f_n form a complete system by themselves.) This corresponds to the nonseparable solution (the so-called diffusion solution) $f_D(x,\mu)=x-g(\mu)$. In fact, the eigenfunctions $f_n(\mu)$ with positive eigenvalues together with $g(\mu)$ form a complete set on the interval where $h(\mu)>0$. These results were conjectured by Fisch and Kruskal [1] and proved by Beals [6] slightly later. This implies that the boundary data at $x=0$ can be expanded as

$$f_+(\mu)=\alpha g(\mu)+\sum_{n\geqslant 0}^{\infty} c_n f_n(\mu). \tag{5}$$

This appears natural since the solutions (3) decay with increasing x. Far from the boundary, one expects the solution to be independent of the details in the boundary condition $f_+(\mu)$. However, unlike the conventional Sturm-Liouville case, the eigenfunctions are not orthogonal and the expansion coefficients are, therefore, not easily calculated. Similarly, the eigenfunctions with negative eigenvalues form a complete but nonorthogonal set on the interval where $h(\mu)<0$, and constitute a natural set of basis functions in which to expand the data on the other boundary, $f_-(\mu)$. The general solution to Eq. (1) can thus be expanded as follows:

$$f(x,\mu)=c_0f_0(\mu)+\alpha f_D(x,\mu)+\sum_{n>0}(c_nf_n(\mu)e^{-\lambda_n x} + c_{-n}f_{-n}(\mu)e^{-\lambda_{-n}(x-L)}). \tag{6}$$

These results suggest that we can construct approximate solutions to the two-way diffusion problem in a natural and systematic way. If L is not too small, one expects that the solution $f(x,\mu)$ can be represented with reasonable accuracy by the first few eigensolutions in most of the domain. The higher-order eigenfunctions have large eigenvalues and are strongly suppressed inside the region. These terms should contribute to the solution only in narrow boundary layers close to the boundaries.

The simplest approximation of $f(x,\mu)$ was determined in a classic paper by Bethe *et al.* [7], and consists in neglecting all the exponentially damped terms in Eq. (6),

$$f(x,\mu)\simeq c_0f_0(\mu)+\alpha f_D(x,\mu)=c_0+\alpha(x-g(\mu)). \tag{7}$$

The problem investigated in Ref. [7] is that of scattering of monoenergetic electrons, which we discuss in detail in the next section. In this problem, $h(\mu)=g(\mu)=\mu$, $D(\mu)=(1-\mu^2)/2$, $a=-1$, $b=1$, and the coefficients c_0 and α were determined by the physical constraint that the approximate solution should describe the correct flux of electrons entering the scattering medium,

$$\int_{h(\mu)>0}f(0,\mu)\mu d\mu=\int_{h(\mu)>0}f_+(\mu)\mu d\mu, \tag{8}$$

$$\int_{h(\mu)<0}f(L,\mu)\mu d\mu=\int_{h(\mu)<0}f_-(\mu)\mu d\mu. \tag{9}$$

In the case of a collimated electron beam with perpendicular incidence, the boundary conditions become

$$f_+(\mu)=\delta(\mu-1), \tag{10}$$

$$f_-(\mu)=0,$$

and we obtain $\alpha=-6/(4+3L)$ and $c=2(1+\alpha/3)$. The total transmitted flux is then given by [7]

$$\Gamma=\int_{-1}^{1}f(x,\mu)\mu d\mu=-\alpha\frac{2}{3}=\frac{1}{1+3L/4}. \tag{11}$$

Given Eq. (6), it is now clear how to improve this zeroth approximation Eq. (7). We include the first eigenfunction and write

$$f(x,\mu)\simeq c_0f_0(\mu)+\alpha f_D(x,\mu)+c_1f_1(\mu)e^{-\lambda_1 x} + c_{-1}f_{-1}(\mu)e^{-\lambda_{-1}(x-L)}. \tag{12}$$

Generalizing the procedure of Bethe *et al.* we determine the coefficients by forming an equation system consisting of Eqs. (8) and (9), written in the more general form

$$\int_{h(\mu)>0}f(0,\mu)h(\mu)d\mu=\int_{h(\mu)>0}f_+(\mu)h(\mu)d\mu, \tag{13}$$

$$\int_{h(\mu)<0}f(L,\mu)h(\mu)d\mu=\int_{h(\mu)<0}f_-(\mu)h(\mu)d\mu, \tag{14}$$

and two further equations involving the next order moments of f, viz.,

$$\int_{h(\mu)>0}f(0,\mu)f_1(\mu)d\mu=\int_{h(\mu)>0}f_+(\mu)f_1(\mu)d\mu, \tag{15}$$

$$\int_{h(\mu)<0}f(L,\mu)f_{-1}(\mu)d\mu=\int_{h(\mu)<0}f_-(\mu)f_{-1}(\mu)d\mu. \tag{16}$$

It is clear that arbitrary accuracy can be obtained by continuing this procedure of adding eigenfunctions to the expansion of f. A consistent scheme for determining the expansion coefficients can now be given by forming an infinite set of suitably chosen weighted averages of the boundary conditions, viz.,

$$\int_{h(\mu)>0}f(0,\mu)w_n^+d\mu=\int_{h(\mu)>0}f_+(\mu)w_n^+d\mu, \tag{17}$$

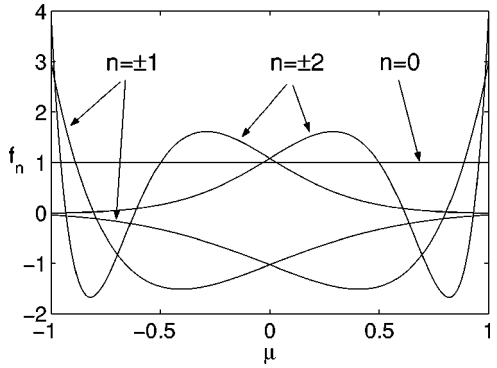

FIG. 1. The numerically calculated eigenfunctions, f_n.

$$\int_{h(\mu)<0} f(L,\mu) w_n^- d\mu = \int_{h(\mu)<0} f_-(\mu) w_n^- d\mu,$$

where the weighting functions are $w_0^+ = w_0^- = h(\mu)$ and $w_n^\pm = f_{\pm n}(\mu)$. When the system is truncated at $n=N$ we obtain a system of $2+2N$ equations for $2+2N$ unknown coefficients $(\alpha, c_0, c_{\pm n})$. This method is quite general, and we demonstrate its potential by considering two particular two-way diffusion problems. The first one is the scattering problem of Bethe *et al.*, and the other one concerns the radiative damping of a relativistic electron beam.

III. EXAMPLE 1: KINETICS OF ELECTRON SCATTERING

Small-angle elastic scattering of particles impinging upon large, stationary scatterers in a slab of length L is described by the following kinetic equation [7]:

$$\mu \frac{\partial f}{\partial x} = \frac{1}{2} \frac{\partial}{\partial \mu} (1-\mu^2) \frac{\partial f}{\partial \mu} \equiv \mathcal{L}(f), \tag{18}$$

where $f = f(x,\mu)$ is the azimuthally integrated steady state distribution function, x is the distance of propagation (normalized to the scattering length), and μ is the cosine of the pitch angle. Unfortunately, the eigenfunctions found from the eigenvalue problem (2),

$$\frac{d}{d\mu}(1-\mu^2)\frac{df_n}{d\mu} + 2\lambda_n \mu f_n = 0, \tag{19}$$

cannot be expressed in elementary functions, but are easily obtained numerically, see Fig. 1. The eigenfunctions are neither even nor odd in μ since the scattering operator $\mathcal{L}$ does not change the parity but multiplication by μ does. For instance, if f is even then $\mathcal{L}(f)$ is even while μf is odd. We also note that there are both positive and negative eigenvalues. Furthermore, if $f_n(\mu)$ is the eigenfunction corresponding to an eigenvalue λ_n, then $f_n(-\mu)$ is the eigenfunction corresponding to the eigenvalue $-\lambda_n$. In particular, $f_0(\mu) = 1$ is an eigenfunction with the eigenvalue $\lambda_0 = 0$.

Even though an exact closed form of the eigenfunctions cannot be given, they can be calculated approximately. One way to do this is by constructing a variational form for the eigenvalue problem and using trial functions to approximate

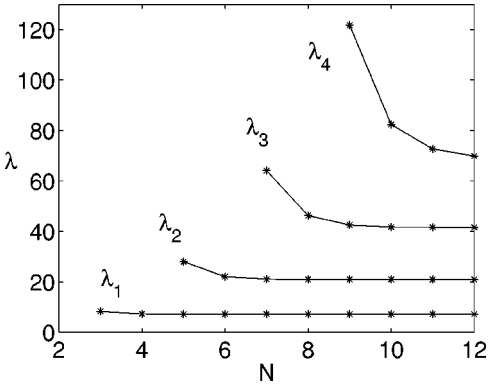

FIG. 2. Convergence of λ_n as a function of the truncation number N.

the eigenfunctions [8]. Another and perhaps more systematic way is to expand the eigenfunctions in Legendre polynomials, $f_n = \Sigma_{l=0}^{\infty} a_l P_l$, as is natural when the diffusion operator is of the form (18). Multiplying Eq. (2) by $P_k(\mu)$ and integrating over μ gives

$$\int_{-1}^{1} P_k \mathcal{L}(f_n) d\mu = -\lambda_n \int_{-1}^{1} \mu P_k f_n d\mu, \tag{20}$$

and by using the orthogonality relation and the fact that $\mathcal{L}(P_l) = -\Sigma_l a_l l(l+1) P_l/2$, this becomes a recurrence equation for the unknown expansion coefficients a_k,

$$\frac{k(k+1)}{2k+1} a_k = \lambda_n \sum_{l=0}^{\infty} a_l \int_{-1}^{1} \mu P_k P_l d\mu. \tag{21}$$

Truncating the Legendre polynomial expansion at the Nth term in Eq. (21) leads to a system of $N+1$ equations, and in order to find nontrivial solutions its coefficient matrix determinant must vanish. This is an equation of even order, and its solutions are the eigenvalues λ_n. As mentioned previously, if λ_n is an eigenvalue so is $-\lambda_n$, and the roots of the determinant equation, therefore, come in pairs $\pm\lambda_n$. The convergence of the eigenvalues with increasing N is shown in Fig. 2. It is seen from this figure that truncating at $N=5$ gives sufficient accuracy for the first eigenfunction, f_1, which then becomes

$$f_1(\mu) \simeq 1 + 1.663\mu + 0.199\mu^2 - 1.868\mu^3 - 2.612\mu^4 - 1.266\mu^5. \tag{22}$$

Comparing this result to that obtained numerically shows that there is no visible difference between the two, Fig. 3.

The coefficients in the first-order expansion of $f(x,\mu)$,

$$f(x,\mu) \simeq c_0(L) f_0 + \alpha(L) f_D(x,\mu) + c_1(L) f_1(\mu) e^{-\lambda_1 x} + c_{-1}(L) f_{-1}(\mu) e^{-\lambda_{-1}(x-L)}, \tag{23}$$

can now be calculated by solving the system of equations (17). The accuracy of this expansion will of course depend on the imposed boundary conditions, and we do not expect this first-order expansion to accurately describe the distribu-

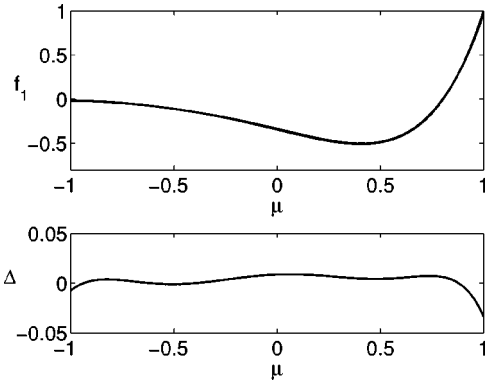

FIG. 3. Comparison between the numerically calculated f_1 and its Legendre polynomial approximation (top). The difference between the two, $\Delta = f_1(legendre) - f_1(num)$ (bottom).

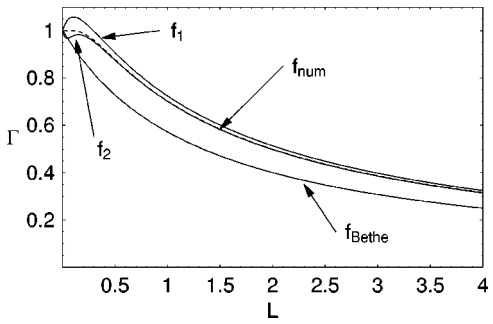

FIG. 4. Transmission flux as a function of slab length L (normalized to the scattering length). f_{num} is the numerically calculated flux, f_{Bethe} is the expression found by Bethe *et al.* [Eq. (11)], f_1 and f_2 are the fluxes found when using the first order, Eq. (23), and second-order approximation of f in calculating α, respectively.

tion function close to the slab surfaces, particularly not when a rapidly varying boundary condition, such as the delta function (10), is used. However, away from the boundary layers, in the bulk of the slab, where the behavior of f is dominated by the diffusion solution f_D, the contributions from the eigenfunctions are small corrections, and here we do expect improvements compared with the zeroth-order expression (7). In order to study these improvements, we examine the particle flux using the same boundary conditions as before, Eq. (10). It is evident that only the diffusion solution f_D should contribute to the flux integral $\Gamma = \int_{-1}^{1} f(x,\mu)\mu d\mu$, since all the higher-order eigenfunctions have vanishing flux. This follows directly from an integration of Eq. (19) over the interval $[-1,1]$. Thus $\alpha(L)$ determines the flux, and we obtain

$$\Gamma = -\frac{2}{3}\alpha(L)$$
$$= -\frac{2}{3}\frac{-2.647+0.520e^{-\lambda_1 L}+0.206e^{-2\lambda_1 L}}{1.439+L-0.085e^{-\lambda_1 L}-(0.073+0.039L)e^{-2\lambda_1 L}}$$
$$\approx \frac{1.774}{1.439+L}, \tag{24}$$

where the last, approximate, equality is obtained by neglecting the exponentials, which is justified if $\lambda_1 L \gg 1$.

Approximate expressions for higher-order eigenfunctions $(f_2, f_3 \ldots)$ can be calculated in the same manner as above. The flux found by including f_2 in the expansion of f is practically exact, see Fig. 4. If a smoother boundary condition is used instead of the delta function, low-order truncations of the expansion for f are even more accurate. Indeed we find that the first-order truncation generally gives a flux with extremely small error, even for very small L.

IV. EXAMPLE 2: DYNAMICS OF RUNAWAY ELECTRONS

Our second problem concerns the kinetics of a relativistic electron beam immersed in a magnetized plasma. This problem is of interest for the theory of ''runaway electrons'' in disrupting tokamak discharges, where fast electrons lose their energy by two mechanisms: by colliding with background plasma particles, and by emitting synchrotron radiation. These two mechanisms are intimately coupled since the emitted power depends on the magnitude and direction of the velocity vector, which continually change because of collisions. The reader is referred to Ref. [5] for a more complete discussion of the relevant physics. Here, we deal with the mathematics of the two-way diffusion problem that arises from the kinetic equation more fully than was done in that paper.

In a plasma immersed in a magnetic field **B**, the kinetic equation for a beamlike $(p_\perp \ll p_\parallel)$ distribution of strongly relativistic electrons experiencing an electric field, Coulomb collisions, and synchrotron radiation reaction is [5]

$$\tau\frac{\partial f}{\partial t} + \left[E - 1 - \frac{\tau}{\tau_r}\left(p_\perp^2 + \frac{\rho_0^2}{R^2}p_\parallel^4\right)\right]\frac{\partial f}{\partial p_\parallel} = \frac{1+Z}{2p_\perp}\frac{\partial}{\partial p_\perp}\left(p_\perp \frac{\partial f}{\partial p_\perp}\right). \tag{25}$$

Here $\mathbf{p} = \gamma\mathbf{v}/c$ is the normalized electron momentum, $\tau = 4\pi\epsilon_0^2 m_e^2 c^3/n_e e^4 \ln\Lambda$ the collision time for relativistic electrons, $\tau_r = 6\pi\epsilon_0(m_e c)^3/e^4 B^2$ the synchrotron radiation time scale, $E = -E_\parallel e\tau/m_e c$ the parallel component of the normalized electric field, $\rho_0 = cm_e/eB$, n_e the plasma electron density, m_e the electron mass, and B the magnetic field strength. The term proportional to E in Eq. (25) thus represents the acceleration caused by the electric field, while the next term (-1) describes friction due to collisions with thermal electrons. The terms involving τ_r represents the reaction force along the magnetic field acting on the electrons as they emit synchrotron radiation. The first one of these terms (proportional to $p_\perp^2$) is the contribution from gyromotion, and the second term that from parallel motion along the magnetic field, whose radius of curvature is taken to be R. The right-hand side of the equation describes scattering of the velocity vector caused by collisions with background electrons and ions of charge Z.

The steady-state distribution in the limit of straight magnetic field $(R \to \infty)$ is thus described by the following two-way diffusion equation:

$$(1-x^2-y^2)\frac{\partial f}{\partial z}=\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}, \tag{26}$$

where we have introduced normalized variables by writing

$$p_\perp^2=\frac{E-1}{\tau/\tau_r}(x^2+y^2),$$

$$p_\parallel=\frac{2\tau_r(E-1)^2 z}{(1+Z)\tau}.$$

The domain where this equation is to be solved is $0<z<\infty$, and the boundary conditions are

$$f(x,y,z)\to 0, \quad x^2+y^2+z^2\to\infty,$$

$$f(x,y,0)=f_+(x,y)=\delta(x)\,\delta(y), \quad x^2+y^2<1. \tag{27}$$

Note that boundary conditions can only be specified on the part of the $z=0$ boundary where $h(x,y)=1-x^2-y^2>0$ because of the two-way nature of the equation. Physically, our choice of a delta function condition on this boundary represents the generation of a runaway electron beam by the electric field if $E>1$ [9]. As these electrons are accelerated, they experience scattering, that increases their perpendicular momentum. For $x^2+y^2>1$, this leads to such large emission of synchrotron radiation that the electrons are slowed down and brought back toward the low-energy boundary $z=0$, which they cross with an unknown distribution at $x^2+y^2>1$.

Although Eq. (26) is formally a two-way diffusion equation in *two* dimensions, our rotationally symmetric boundary condition makes the problem effectively one-dimensional, and the theory from Sec. II applies. A more fundamental difference between this problem and that considered in the previous sections is that there is no diffusion solution of Eq. (26) satisfying the boundary conditions since there is now no solution of Eq. (4) that vanishes at infinity. The solution of Eq. (26) can, therefore, be represented purely by a linear combination of eigenfunctions

$$\frac{\partial^2 f_{kl}}{\partial x^2}+\frac{\partial^2 f_{kl}}{\partial y^2}=-\gamma_{kl}(1-x^2-y^2)f_{kl}.$$

In contrast to Sec. III, these functions can be expressed analytically [5],

$$f_{kl}(x,y)=\exp\left(-\frac{\sqrt{\gamma_{kl}}}{2}(x^2+y^2)\right)H_k(x\gamma_{kl}^{1/4})H_l(y\gamma_{kl}^{1/4}), \tag{28}$$

in terms of Hermite polynomials H_k, and the eigenvalues are

$$\gamma_{kl}=4(k+l+1)^2, \quad k,l=0,1,\ldots. \tag{29}$$

Alternatively, Eq. (26) can also be separated in cylindrical coordinates. This leads to an eigenvalue problem equivalent to the two-dimensional harmonic oscillator in quantum mechanics, which can be solved in terms of Laguerre polynomials [10]. The distribution function expansion (6) becomes

$$f(x,y,z)=\sum_{k,l=0}^{\infty} c_{kl}f_{kl}(x,y)\exp(-\gamma_{kl}z), \tag{30}$$

and we observe that for large z the dominant behavior of f is found from the eigenfunction with the lowest eigenvalue since the terms decay exponentially, $\propto\exp(-\gamma_{kl}z)$. The expansion coefficients, c_{kl}, are determined by solving the system of equations (17),

$$\int_{h(x,y)>0} f(x,y,0)w_{kl}^+ dx\,dy=\int_{h(x,y)>0} f_+(x,y)w_{kl}^+ dx\,dy, \tag{31}$$

with weighting functions $w_{kl}^+=f_{kl}$. $h(\mu)$ has been excluded from the weighting functions since there is no diffusion solution to this problem. Since the boundary condition $f_+(x,y)=\delta(x)\,\delta(y)$ at $z=0$ is even in x and y, the coefficients c_{kl} vanish when either index is odd. The nonzero coefficients decrease rapidly with increasing indices. By numerically solving Eq. (31), we find that c_{00} converges towards 1.19 and the lowest-order approximation of f valid for $z\gg 1/\gamma_{02}=1/36$ thus becomes

$$\begin{aligned} f(x,y,z)&\approx c_{00}f_{00}(x,y)\exp(-\gamma_{00}z)\\ &=1.19\exp[-(x^2+y^2)-4z]. \end{aligned} \tag{32}$$

When considering smaller z, higher-order approximations are needed to describe f accurately. The coefficients in the next four approximations are calculated numerically and found to be

$$c_{02}=c_{20}=-0.97,$$

$$c_{04}=c_{40}=0.198, c_{22}=2c_{04},$$

$$c_{06}=c_{60}=-0.0232, c_{24}=c_{42}=3c_{06},$$

$$c_{08}=c_{80}=0.0019, c_{26}=c_{62}=4c_{08}, c_{44}=6c_{08}.$$

Including these terms gives an accurate description of f everywhere except at very small z. In this region, $z\to 0$, the boundary condition (27) says nothing about the distribution function for $x^2+y^2>1$, but dictates that it should be strongly peaked around $x=y=0$ for $x^2+y^2<1$. In fact, for $r^2=x^2+y^2\ll 1$, our Eq. (26) reduces to an ordinary diffusion equation, with the solution

$$f(r,z)=\frac{1}{4\pi z}\exp\left(-\frac{r^2}{4z}\right). \tag{33}$$

Figures 5 and 6 show the truncated series solution (30) at $z=0.015$ and $z=0.06$, together with the asymptotic expressions (32) and (33). For small r the solution closely resembles Eq. (33), and for large r and large z it approaches Eq. (32). The two-way diffusion nature of the equation is also evident in Fig. 5: there is a population of backscattered electrons in the region $r\sim 1.2$.

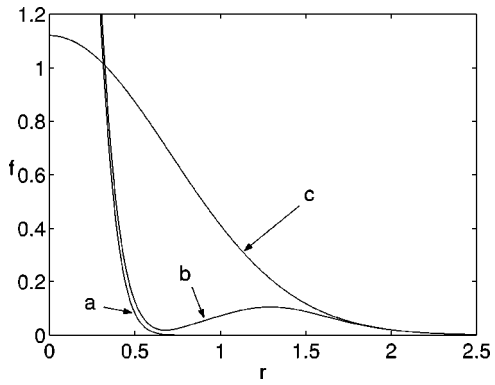

FIG. 5. Distribution function f vs $r=(x^2+y^2)^{1/2}$ at $z=0.015$: (a) one-way diffusion solution (33); (b) series solution (30) truncated after terms with $k+l=8$; and (c) zeroth-order approximation, Eq. (32).

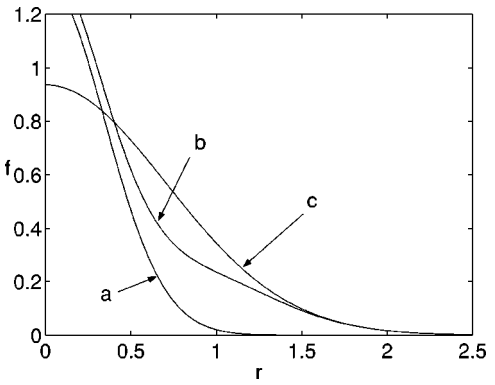

FIG. 6. Same as Fig. 5 but for $z=0.06$.

V. SUMMARY

It is known from the mathematical literature that two-way diffusion equations have the property that boundary conditions can only be prescribed on ''half'' the boundaries, and that ''half'' of the eigenfunctions form a complete but nonorthogonal set on these ''half'' boundaries. In the present paper, we have shown how these properties can be used to construct approximate solutions to these equations in a systematic way. The power of the method is demonstrated by examining two particular two-way diffusion problems: scattering of electrons by stationary particles in a slab; and damping of a relativistic electron beam by the combined action of scattering and synchrotron radiation emission. The expansion converges quickly in both these problems, even if the mean-free path is not very short, and it, therefore, appears that the method should be of wide applicability.

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